Description of random radiating and nonradiating sources

Edwin A. Marengo and Richard W. Ziolkowski
Department of Electrical and Computer Engineering, The University of Arizona, Tucson, Arizona 85721
(Received 28 January 2000)

An alternative description of spatially localized random nonradiating sources is derived, for any degree of spatial-temporal coherence. A description of a complementary class of random sources (purely radiating sources) that lacks a nonradiating part is also obtained.

PACS number(s): 42.25.Kb, 41.20.−q

A number of results reported in a recent paper [1], to be referred to as I, are applied to derive a description of spatially localized random nonradiating (NR) sources and of random sources that lack a NR part (purely radiating sources). Both NR sources and purely radiating sources that lack a NR part play a fundamental role in inverse problems. For example, the class of purely radiating sources includes the so-called minimum energy solutions to the (inverse source) problem of reconstructing an unknown source from knowledge of its exterior field [2]. NR sources are also of interest in semi-classical models of electrons and atoms [3]. Our results are presented in the framework of the scalar wave equation in three-dimensional (3D) free space for sources of any degree of spatial-temporal coherence. They are readily generalized to source-field systems governed by other partial differential equations (PDEs), including electromagnetic source-field systems in arbitrary linear media.

In I we considered a general complex-valued scalar, vector, or tensor source-field system described (in shorthand notation) by a linear PDE $L\psi(\mathbf{x})=\rho(\mathbf{x})$, where ψ, ρ, and L represent, respectively, the associated field, source and partial differential operator (PDO) in a configuration space $\mathbf{x}\in\mathbb{R}^n$. The field ψ produced by a source ρ can be expressed as $\psi(\mathbf{x})=\int d^nx'\,\rho(\mathbf{x}')G(\mathbf{x}|\mathbf{x}')$, where G is the Green function associated with the PDO L and the given boundary conditions. For a NR source $\rho_{\rm NR}$ of support D, the generated field $\psi(\mathbf{x})=0$ if $\mathbf{x}\notin D$. It was shown in I that a scalar, vector, or tensor source $\rho_{\rm NR}$ of support D is NR if and only if it obeys the orthogonality relation

$$\int_D d^nx\, v^*(\mathbf{x})\rho_{\rm NR}(\mathbf{x})=0, \tag{1}$$

where the asterisk denotes the complex conjugate, with respect to all solutions $v(\mathbf{x})$ of the homogeneous form of the associated adjoint PDE $\tilde{L}v(\mathbf{x})=0$ for $\mathbf{x}\in D$, where $\tilde{L}$ is the adjoint of the PDO L (as defined, e.g., in Ref. [4]). For a (formally) self-adjoint PDO, $\tilde{L}=L$ (such as the PDO's of the usual acoustic and electromagnetic fields), and this requirement becomes $Lv(\mathbf{x})=0$ for $\mathbf{x}\in D$. The reciprocity principle draws the following physical picture for this result: if a source does not radiate, then it does not receive either, and vice versa; i.e., NR sources are both invisible to external observers and noninteracting to external fields. Furthermore, it was also shown in I that, for self-adjoint PDOs L, the condition

$$L\rho(\mathbf{x})=0 \quad \text{if } \mathbf{x}\in D \ \text{(the boundary } \partial D \text{ of } D \text{ excluded)} \tag{2}$$

is both necessary and sufficient for a square-integrable (L_2) source ρ of support D to lack a NR part such that $\int_D d^nx\,\rho_{\rm NR}^*(\mathbf{x})\rho(\mathbf{x})=0$ for all $L_2(D)$ NR sources $\rho_{\rm NR}$. One naturally extends this result to the inverse source problem, concluding that the familiar minimum energy solution, which corresponds to the radiating portion of the unknown source (as shown in Ref. [5] and, more recently, in Ref. [6]) must be, itself, a free field, truncated within the source support. That this is, in fact, the case, can be verified, for special cases, from other studies [2,6,7]; this includes the vector case [8]. One also deduces from this result and the unique decomposition of any $L_2(D)$ source into the sum of a radiating part and a NR part (see Ref. [9], pp. 12 and 13) that, ultimately, the radiating parts, i.e., the sources of radiation, are, themselves, fields. These general results, plus a number of related results corresponding to special cases, were presented in Refs. [1,6,7] in connection with deterministic sources. It remains to explain their random source aspects.

In order to explain how these results apply not only to deterministic sources but also to random sources, we next consider scalar, random source-field systems whose source-field realizations (q,ψ) are described in the 4D space-frequency domain by the inhomogeneous Helmholtz equation $(\nabla^2+k^2)\psi(\mathbf{r},\omega)=q(\mathbf{r},\omega)$, where $k=\omega/c$ is the wave number of the field at the temporal frequency ω. In the most general case, a scalar, random source and its field are described in the space-frequency domain by correlations of order (m,n), i.e. [10],

$$Q^{(m,n)}(\chi)=\left\langle \prod_{j=1}^{m} q^*(\mathbf{r}_j,\omega_j)\prod_{k=m+1}^{m+n} q(\mathbf{r}_k,\omega_k)\right\rangle,$$
$$\Psi^{(m,n)}(\chi)=\left\langle \prod_{j=1}^{m} \psi^*(\mathbf{r}_j,\omega_j)\prod_{k=m+1}^{m+n} \psi(\mathbf{r}_k,\omega_k)\right\rangle, \tag{3}$$

where we have introduced the shorthand notation $\chi\equiv\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_{m+n};\omega_1,\omega_2,\ldots,\omega_{m+n}$, and where the angular brackets denote ensemble averages. Also, in the following, $\chi'\equiv\mathbf{r}'_1,\mathbf{r}'_2,\ldots,\mathbf{r}'_{m+n};\omega_1,\omega_2,\ldots,\omega_{m+n}$. In this general $(m,\ n)$ order framework, the field correlation $\Psi^{(m,n)}$, corresponding to a source correlation $Q^{(m,n)}$, is the solution [10]

 ©2000 The American Physical Society

$$\Psi^{(m,n)}(\chi)=\left[\prod_{j=1}^{m}\int_D d^3r_j'\,G^*(\mathbf{r}_j|\mathbf{r}_j';\omega_j)\right]\times\left[\prod_{k=m+1}^{m+n}\int_D d^3r_k'\,G(\mathbf{r}_k|\mathbf{r}_k';\omega_k)\right]Q^{(m,n)}(\chi')\tag{4}$$

of the PDE,

$$L^{(m,n)}\Psi^{(m,n)}(\chi)=Q^{(m,n)}(\chi),\tag{5}$$

where $L^{(m,n)}$ is the self-adjoint PDO defined by

$$L^{(m,n)}=\prod_{j=1}^{m+n}(\nabla_j^2+k_j^2),\tag{6}$$

where ∇_j^2 denotes the Laplacian operator taken with respect to $\mathbf{r}_j$. In Eq. (4), G is the outgoing Green function of the free space Helmholtz operator. The special case $m=n=1$ corresponds to the usual second-order theory. In this special case, expressions (5) and (6) reduce to [10]

$$(\nabla_1^2+k_1^2)(\nabla_2^2+k_2^2)\Psi^{(1,1)}(\mathbf{r}_1,\mathbf{r}_2;\omega_1,\omega_2)=Q^{(1,1)}(\mathbf{r}_1,\mathbf{r}_2;\omega_1,\omega_2).\tag{7}$$

Moreover, if the statistical ensemble that represents the source is temporally stationary (at least in the wide sense), then expression (7) takes the familiar restricted form [10]

$$(\nabla_1^2+k^2)(\nabla_2^2+k^2)\Psi^{(1,1)}(\mathbf{r}_1,\mathbf{r}_2;\omega)=Q^{(1,1)}(\mathbf{r}_1,\mathbf{r}_2;\omega).\tag{8}$$

With these equations, we are in position to apply the general results of I to random sources of arbitrary degree of spatial-temporal coherence.

We note that, in the (*m*, *n*) order description, a scalar, random source of spatial support D is defined to be NR if its generated field correlation $\Psi^{(m,n)}(\chi)=0$ if $\mathbf{r}_j\notin D$, for any $j=1,2,\ldots,m+n$ [11–13]. We now conclude from Eqs. (5) and (6), and the statement given in connection with Eq. (1), that, *in the (m, n) order description, a necessary and sufficient condition for a localized, random source of spatial support D to be NR is that*

$$\int_D d^3r_1\cdots\int_D d^3r_{m+n}(V^{(m,n)})^*(\chi)Q_{\mathrm{NR}}^{(m,n)}(\chi)=0,\tag{9}$$

where $Q_{\mathrm{NR}}^{(m,n)}$ *is the NR source correlation of order (m, n), and* $V^{(m,n)}(\chi)$ *is any solution of the homogeneous PDE* $L^{(m,n)}V^{(m,n)}(\chi)=0$ *for* $\mathbf{r}_j\in D$, *for all* $j=1,2,\ldots,m+n$. Here, for the source-field system $(Q^{(m,n)},\Psi^{(m,n)})$, the PDO $L^{(m,n)}$ thus plays the role of the generic PDO L of the general theory. For the usual second-order description, associated with expressions (7) and (8), this result reduces to the following: *A necessary and sufficient condition for a localized, random source whose realizations are characterized by a (second-order) correlation* $Q_{\mathrm{NR}}^{(1,1)}$ *to be NR is that* $Q_{\mathrm{NR}}^{(1,1)}$ *be orthogonal to all solutions of the homogeneous form of Eq. (7) [or, for wide sense stationary sources, the homogeneous form of Eq. (8)] for* $\mathbf{r}_j\in D$, *for all* $j=1,2$. Furthermore, we also conclude from Eqs. (5) and (6) and the statement given in connection with Eq. (2) that *a necessary and sufficient condition for an* L_2 *random source of support D whose realizations are characterized by an (m, n) order correlation* $Q^{(m,n)}$ *to lack a NR part is that* $Q^{(m,n)}$ *obeys*

$$L^{(m,n)}Q^{(m,n)}(\chi)=0\quad\text{for }\mathbf{r}_j\in D\ (\text{the boundary }\partial D\text{ of }D\text{ excluded}),\ \text{for all }j=1,2,\ldots,m+n.\tag{10}$$

Again, result (10) is important for minimum energy-constrained formulations of inverse problems with random sources [the validity of Eq. (10) can be verified for special cases considered in Refs. [2,14,15]].

The previously unknown statements given in connection with Eqs. (9) and (10) establish a characterization of these two complementary classes of spatially localized random sources, i.e., random NR sources and purely radiating random sources. Physically, these results are manifestations of the reciprocity property, as we mentioned earlier. Further manipulations and related results for the associated deterministic case can be found in Refs. [1,6,7], and are extended readily to the random case with the above-provided expressions.

Since all of the preceding scalar random results follow directly from the general radiating NR source theory presented in I, their validity rests on the validity of the more general results which was established in I. Even so, for the sake of completeness and, in particular, in order to provide the full random picture of the general results, we consider next two alternative ways of understanding the results (9) and (10).

The definition of a general scalar, vector, or tensor localized NR source given in connection with Eq. (1) was derived in I by means of the generalized Green theorem and the familiar Devaney-Wolf representation of a localized NR source [16] generalized to any source-field system. In the present context, which focuses on scalar, random sources in free space, one can use an analogous method to verify the validity of Eq. (9) and the associated discussion. In particular, first we note that the most general localized NR source correlation $Q^{(m,n)}$ of order (*m*,*n*) must be expressible as [11,12]

$$Q_{\mathrm{NR}}^{(m,n)}(\chi)=L^{(m,n)}U_0^{(m,n)}(\chi),\tag{11}$$

where the localized function $U_0^{(m,n)}(\chi)=0$ if $\mathbf{r}_j\notin D$, for any $j=1,2,\ldots,m+n$ (where, again, D is the support of the NR source realizations), whereas

$$U_0^{(m,n)}(\chi)=\left[\prod_{j=1}^{m}\int_D d^3r_j'\,G^*(\mathbf{r}_j|\mathbf{r}_j';\omega_j)\right]\times\left[\prod_{k=m+1}^{m+n}\int_D d^3r_k'\,G(\mathbf{r}_k|\mathbf{r}_k';\omega_k)\right]Q_{\mathrm{NR}}^{(m,n)}(\chi')$$

if $\mathbf{r}_j\in D$, for all $j=1,2,\ldots,m+n$. It follows from Eq. (4) that $U_0^{(m,n)}$ is exactly the NR field correlation produced by this localized NR source correlation. It can be shown from

Eq. (11) and the generalized Green theorem that for any localized NR source correlation,

$$\int_D d^3r_1 \cdots \int_D d^3r_{m+n} (V^{(m,n)})^*(\chi) Q_{\rm NR}^{(m,n)}(\chi)$$
$$= \int_D d^3r_1 \cdots \int_D d^3r_{m+n} [L^{(m,n)} V^{(m,n)}]^*(\chi) U_0^{(m,n)}(\chi), \tag{12}$$

where the function $V^{(m,n)}$ is arbitrary. However, the integral in Eq. (12) is seen to vanish if $L^{(m,n)}V^{(m,n)}(\chi)=0$ for $\mathbf{r}_j \in D$, for all $j=1,2,\ldots,m+n$. This provides the necessary portion of the NR source correlation condition given in connection with Eq. (9). To show sufficiency (for a localized, random source to be NR), one simply notes that

$$L^{(m,n)} \prod_{j=1}^{m} G(\mathbf{r}_j'|\mathbf{r}_j;\omega_j) \prod_{k=m+1}^{m+n} G^*(\mathbf{r}_k'|\mathbf{r}_k;\omega_k) = 0$$
$$\text{if } \mathbf{r}_j' \notin D \quad \text{and } \mathbf{r}_j \in D, \quad \text{for any } j=1,2,\ldots,m+n$$

(which holds, in fact, for any Green function G of the Helmholtz operator), so that, by using the orthogonality condition in Eq. (9), the field correlation produced by $Q_{\rm NR}^{(m,n)}$ [as defined by Eq. (4)]

$$\left[\prod_{j=1}^{m} \int_D d^3r_j\, G^*(\mathbf{r}_j'|\mathbf{r}_j;\omega_j)\right] \times \left[\prod_{k=m+1}^{m+n} \int_D d^3r_k\, G(\mathbf{r}_k'|\mathbf{r}_k;\omega_k)\right] Q_{\rm NR}^{(m,n)}(\chi) = 0$$
$$\text{if } \mathbf{r}_j' \notin D \quad \text{for any } j=1,2,\ldots,m+n,$$

which completes the proof.

For a general localized, random NR source of support D [not necessarily an $L_2(D)$ random NR source], the integral in Eq. (12) involves the entire support D (including the boundary ∂D of D). For the special case of an $L_2(D)$ random NR source, on the other hand, the integral in Eq. (12) involves only the interior of D (the boundary ∂D of D excluded). In particular, for that special case, the contribution of the boundary ∂D to the integral must vanish due to the vanishing of the NR field realizations over ∂D, as follows from the so-called NR boundary conditions for the Helmholtz operator [17–19]; for a localized, random NR source and, in particular, for the PDO $L^{(m,n)}$, they require the vanishing of the NR field correlation $U_0^{(m,n)}(\chi)$ and of its first partial derivatives for $\mathbf{r}_j \in \partial D$, for all $j=1,2,\ldots,m+n$. With this clarification, we also see from the preceding manipulations and, in particular, expression (12), that, in the (m,n) order description, an $L_2(D)$ random source will lack NR source realizations if and only if its source correlation $Q^{(m,n)}(\chi)$ obeys the homogeneous PDE (10), since this and only this ensures the vanishing of the (orthogonality) integral in Eq. (12) for any choice of the NR field correlation $U_0^{(m,n)}$ subjected to the aforementioned NR boundary constraints. This completes the Green-function-based picture of the results associated with Eqs. (9) and (10). We consider next a related point of view that is based on the analyticity of far field correlations, and which complements the preceding Green-function-based formalism.

The (m,n) order far field correlation

$$F^{(m,n)}(\xi) = \left\langle \prod_{j=1}^{m} f^*(\mathbf{s}_j,\omega_j) \prod_{k=m+1}^{m+n} f(\mathbf{s}_k,\omega_k) \right\rangle,$$

where $\xi \equiv \mathbf{s}_1,\mathbf{s}_2,\ldots,\mathbf{s}_{m+n};\omega_1,\omega_2,\ldots,\omega_{m+n}$, corresponding to a given source correlation $Q^{(m,n)}(\chi)$, is defined by [11,12]

$$F^{(m,n)}(\xi) = \int_D d^3r_1 \cdots \int_D d^3r_{m+n} Q^{(m,n)}(\chi) \times \exp[i(k_1\mathbf{s}_1\cdot\mathbf{r}_1 + \cdots + k_m\mathbf{s}_m\cdot\mathbf{r}_m - k_{m+1}s_{m+1}\cdot\mathbf{r}_{m+1} - \cdots - k_{m+n}\mathbf{s}_{m+n}\cdot\mathbf{r}_{m+n})],$$

which is identified to be the value of the $3(m+n)$-fold spatial Fourier transform $\tilde{Q}^{(m,n)}(\mathbf{K}_1,\mathbf{K}_2,\ldots,\mathbf{K}_{m+n})$ of $Q^{(m,n)}(\chi)$ evaluated at $\mathbf{K}_1=-k_1\mathbf{s}_1,\ldots,\mathbf{K}_m=-k_m\mathbf{s}_m, \mathbf{K}_{m+1}=k_{m+1}s_{m+1},\ldots,\mathbf{K}_{m+n}=k_{m+n}\mathbf{s}_{m+n}$. For a localized, random source correlation, the above $3(m+n)$-fold Fourier transform is an entire analytic function on the $3(m+n)$ Fourier variables (as required by the Plancherel-Polya theorem [15]), so that the far field correlations are, themselves, analytic on the observation unit vectors $\mathbf{s}_1,\mathbf{s}_2,\ldots,\mathbf{s}_{m+n}$ (also see Ref. [16] for the deterministic version of this well-known result). It is also known that a necessary and sufficient condition for a localized NR source correlation $Q_{\rm NR}^{(m,n)}$ to be NR is that its far field correlation [11–13]

$$F^{(m,n)}(\xi) = \int_D d^3r_1 \cdots \int_D d^3r_{m+n} Q_{\rm NR}^{(m,n)}(\chi) \times \exp[i(k_1\mathbf{s}_1\cdot\mathbf{r}_1 + \cdots + k_m\mathbf{s}_m\cdot\mathbf{r}_m - k_{m+1}\mathbf{s}_{m+1}\cdot\mathbf{r}_{m+1} - \cdots - k_{m+n}\mathbf{s}_{m+n}\cdot\mathbf{r}_{m+n})] = 0 \tag{13}$$

for all real observation unit vectors $\mathbf{s}_1,\mathbf{s}_2,\ldots,\mathbf{s}_{m+n}$. Now, because of the analyticity of the far field correlation $F^{(m,n)}(\xi)$, it follows that the NR source correlation condition (13) must hold not only for all real but also for all complex observation unit vectors $\mathbf{s}_1,\mathbf{s}_2,\ldots,\mathbf{s}_{m+n}$. Therefore, for a localized random NR source correlation, the entire propagating plus evanescent plane wave spectrum must vanish (consequently, the complete external field correlation must vanish, as has been known for a long time [16]). In addition, expression (13) is, by itself, an interaction integral, so that the same NR statement given in connection with Eq. (13) also draws the associated noninteracting picture: a localized random NR source of support D neither radiates nor receives any homogeneous or evanescent plane wave. Moreover, since the field produced by any source external to D must be, itself, entirely expressible as an angular spectrum of homogeneous and evanescent plane waves, it follows that a localized random NR source must not receive (interact with) any field produced outside D. The analyticity property of the (external field) plane wave spectrum thus explains both the localizability of the NR field correlation (within the random NR source support) and the noninteractivity of a localized NR source correlation to all incident field correlations (both homogeneous and evanescent plane wave correlations and, consequently, any externally produced field correlation). We have thus complemented the Green-function-based manipulations by furnishing an alternative interpretation of our localized NR source correlation results that is based on the analyticity of far field correlations.

Finally, we wish to mention that all of the preceding scalar random results can be extended readily to the electromagnetic case by substituting the scalar PDO $\nabla_j^2+k_j^2$ in the above expressions [in particular, in the definitions given in connection with Eqs. (9) and (10)] by the vector PDO $\boldsymbol{\nabla}_j \times\boldsymbol{\nabla}_j\times -k_j^2$ (also see Ref. [1] for the corresponding deterministic case). For example, if follows from result (9) and the associated discussion that *a necessary and sufficient condition for a localized, electromagnetic random source* $j_{\rm NR}$ *characterized in the space-frequency domain by a second-order correlation tensor*

$$\mathcal{J}_{\rm NR}^{(1,1)}(\mathbf{r}_1,\mathbf{r}_2;\omega_1,\omega_2)=\langle \mathbf{j}_{\rm NR}^*(\mathbf{r}_1,\omega_1)\mathbf{j}_{\rm NR}(\mathbf{r}_2,\omega_2)\rangle$$

(*in which* $\mathbf{j}_{\rm NR}^*(\mathbf{r}_1,\omega_1)\mathbf{j}_{\rm NR}(\mathbf{r}_2,\omega_2)$ *is read as a dyadic product) to be NR is that* $\mathcal{J}_{\rm NR}^{(1,1)}$ be orthogonal to all solutions of the homogeneous PDE:

$$(\boldsymbol{\nabla}_1\times\boldsymbol{\nabla}_1\times-k_1^2)(\boldsymbol{\nabla}_2\times\boldsymbol{\nabla}_2\times-k_2^2)\mathcal{V}^{(m,n)}(\mathbf{r}_1,\mathbf{r}_2;\omega_1,\omega_2)$$
$$=0 \quad \text{for } \mathbf{r}_j\in D \quad \text{for all } j=1,2. \tag{14}$$

The more general electromagnetic (m,n) order description follows obvious lines. Extension of the result in Eq. (10) to the electromagnetic case is also obvious, and yields a description of all such purely radiating vector sources.

Summarizing, in this paper we developed a description of spatially localized random NR sources and random sources that lack a NR part: NR source correlations are, by definition, orthogonal to all correlations that behave like free fields in the NR source support, whereas purely radiating source correlations are, themselves, free-field correlations, truncated within the source support. Our results on random NR sources lead to previously known results on such sources reported, e.g., in Refs. [2,11–15]. In particular, all the definitions of localized NR sources given in those studies involved orthogonality relations of the NR sources and free fields, e.g., plane waves in the far-field-based definitions and source-free multipoles in the near-field-based descriptions. Analogous observations apply to the purely radiating, minimum energy sources presented in Refs. [2,14,15], all of which are seen to consist of free fields truncated within the source support. Elsewhere we plan to apply the results of this paper to inverse problems with random sources.

This work was supported by the National Science Foundation under Grant No. ECS-9900246, and by the Air Force Office of Scientific Research, Air Force Materials Command, USAF, under Grant No. F49620-96-1-0039.

[1] E. A. Marengo and R. W. Ziolkowski, Phys. Rev. Lett. **83**, 3345 (1999).

[2] I. J. LaHaie, J. Opt. Soc. Am. A **2**, 35 (1985).

[3] G. H. Goedecke, Phys. Rev. **135**, B281 (1964).

[4] G. Arfken, *Mathematical Methods for Physicists* (Academic, San Diego, 1985).

[5] A. J. Devaney and R. P. Porter, J. Opt. Soc. Am. A **2**, 2006 (1985).

[6] E. A. Marengo, A. J. Devaney, and R. W. Ziolkowski, J. Opt. Soc. Am. A **17**, 34 (2000).

[7] E. A. Marengo, A. J. Devaney, and R. W. Ziolkowski, J. Opt. Soc. Am. A **16**, 1612 (1999).

[8] E. A. Marengo and A. J. Devaney, IEEE Trans. Antennas Propag. **47**, 410 (1999).

[9] M. Bertero, in *Advances in Electronics and Electron Physics*, edited by P. W. Hawkes (Academic, San Diego, 1989), Vol. 75, pp. 1–120.

[10] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, New York, 1995).

[11] B. J. Hoenders and H. P. Baltes, Lett. Nuovo Cimento Soc. Ital. Fis. **25**, 206 (1979).

[12] B. J. Hoenders and H. P. Baltes, J. Phys. A **13**, 995 (1980).

[13] A. J. Devaney and E. Wolf, in *Coherence and Quantum Optics*, edited by L. Mandel and E. Wolf (Plenum, New York, 1984), pp. 417–421.

[14] I. J. LaHaie, J. Opt. Soc. Am. A **3**, 1073 (1986).

[15] A. J. Devaney, J. Math. Phys. **20**, 1687 (1979).

[16] A. J. Devaney and E. Wolf, Phys. Rev. D **8**, 1044 (1973).

[17] A. Gamliel, K. Kim, A. I. Nachman, and E. Wolf, J. Opt. Soc. Am. A **6**, 1388 (1989).

[18] A. J. Devaney and E. A. Marengo, Pure Appl. Opt. **7**, 1213 (1998).

[19] E. A. Marengo and R. W. Ziolkowski, J. Math. Phys. **41**, 845 (2000).